12

Dynamic Analysis and Response of Linear Systems

PREVIEW

Now that we have developed procedures to formulate the equations of motion for MDF systems subjected to dynamic forces (Chapters 9 and 11), we are ready to present the solution of these equations. In Part A of this chapter we show that the equations for a two-DOF system without damping subjected to harmonic forces can be solved analytically. Then we use these results to explain how a vibration absorber or tuned mass damper works to decrease or eliminate unwanted vibration. This simultaneous solution of the coupled equations of motion is not feasible in general, so in Part B we develop the classical modal analysis procedure. The equations of motion are transformed to modal coordinates, leading to an uncoupled set of modal equations; each modal equation is solved to determine the modal contributions to the response, and these modal responses are combined to obtain the total response. An understanding of the relative response contributions of the various modes is developed in Part C with the objective of deciding the number of modes to include in dynamic analysis. The chapter closes with Part D, which includes two analysis procedures useful in special situations: static correction method and mode acceleration method.

PART A: TWO-DEGREE-OF-FREEDOM SYSTEMS

12.1 ANALYSIS OF TWO-DOF SYSTEMS WITHOUT DAMPING

Consider the two-DOF systems shown in Fig. 12.1.1 excited by a harmonic force $p_1(t) = p_o \sin \omega t$ applied to the mass m_1. For both systems the equations of motion are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_o \\ 0 \end{Bmatrix} \sin \omega t \tag{12.1.1}$$

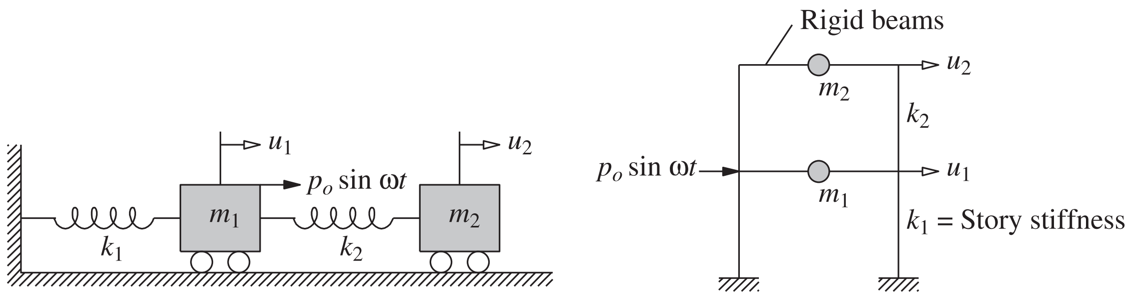

Figure 12.1.1 Two-degree-of-freedom systems.

Observe that the equations are coupled through the stiffness matrix. One equation cannot be solved independent of the other; that is, both equations must be solved simultaneously. Because the system is undamped, the steady-state solution can be assumed as

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} \sin \omega t$$

Substituting this into Eq. (12.1.1), we obtain

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\,\omega^2 \end{bmatrix} \begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} = \begin{Bmatrix} p_o \\ 0 \end{Bmatrix} \tag{12.1.2}$$

or

$$\left[\mathbf{k} - \omega^2\mathbf{m}\right] \begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} = \begin{Bmatrix} p_o \\ 0 \end{Bmatrix}$$

Premultiplying by $[\mathbf{k} - \omega^2\mathbf{m}]^{-1}$ gives

$$\begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} = \left[\mathbf{k} - \omega^2\mathbf{m}\right]^{-1} \begin{Bmatrix} p_o \\ 0 \end{Bmatrix} = \frac{1}{\det\,[\mathbf{k} - \omega^2\mathbf{m}]} \operatorname{adj}\,[\mathbf{k} - \omega^2\mathbf{m}] \begin{Bmatrix} p_o \\ 0 \end{Bmatrix} \tag{12.1.3}$$

where det[·] and adj[·] denote the determinant and adjoint of the matrix[·], respectively. The frequency equation [Eq. (10.2.6)]

$$\det[\mathbf{k} - \omega^2\mathbf{m}] = 0$$

can be solved for the natural frequencies ω_1 and ω_2 of the system. In terms of these frequencies, this determinant can be expressed as

$$\det[\mathbf{k} - \omega^2\mathbf{m}] = m_1 m_2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) \tag{12.1.4}$$

Thus, Eq. (12.1.3) becomes

$$\begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} = \frac{1}{\det[\mathbf{k} - \omega^2\mathbf{m}]} \begin{bmatrix} k_2 - m_2\omega^2 & k_2 \\ k_2 & k_1 + k_2 - m_1\omega^2 \end{bmatrix} \begin{Bmatrix} p_o \\ 0 \end{Bmatrix} \tag{12.1.5}$$

or

$$u_{1o} = \frac{p_o(k_2 - m_2\omega^2)}{m_1 m_2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \qquad u_{2o} = \frac{p_o k_2}{m_1 m_2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \tag{12.1.6}$$

Example 12.1

Plot the frequency-response curve for the system shown in Fig. 12.1.1 with $m_1 = 2m$, $m_2 = m$, $k_1 = 2k$, and $k_2 = k$ subjected to harmonic force p_o applied on mass m_1.

Solution Substituting the given mass and stiffness values in Eq. (12.1.6) gives

$$u_{1o} = \frac{p_o(k - m\omega^2)}{2m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \qquad u_{2o} = \frac{p_o k}{2m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \tag{a}$$

where $\omega_1 = \sqrt{k/2m}$ and $\omega_2 = \sqrt{2k/m}$; these natural frequencies were obtained in Example 10.4. For given system parameters, Eq. (a) provides solutions for the response amplitudes u_{1o} and u_{2o}. It is instructive to rewrite them as

$$\frac{u_{1o}}{(u_{1\text{st}})_o} = \frac{1 - \frac{1}{2}(\omega/\omega_1)^2}{\left[1 - (\omega/\omega_1)^2\right]\left[1 - (\omega/\omega_2)^2\right]} \qquad \frac{u_{2o}}{(u_{2\text{st}})_o} = \frac{1}{\left[1 - (\omega/\omega_1)^2\right]\left[1 - (\omega/\omega_2)^2\right]} \tag{b}$$

In these equations the response amplitudes have been divided $(u_{1\text{st}})_o = p_o/2k$ and $(u_{2\text{st}})_o = p_o/2k$, the maximum values of the *static displacements* (a concept introduced in Section 3.1), to obtain normalized or nondimensional responses that depend on frequency ratios ω/ω_1 and ω/ω_2, not separately on ω, ω_1, and ω_2.

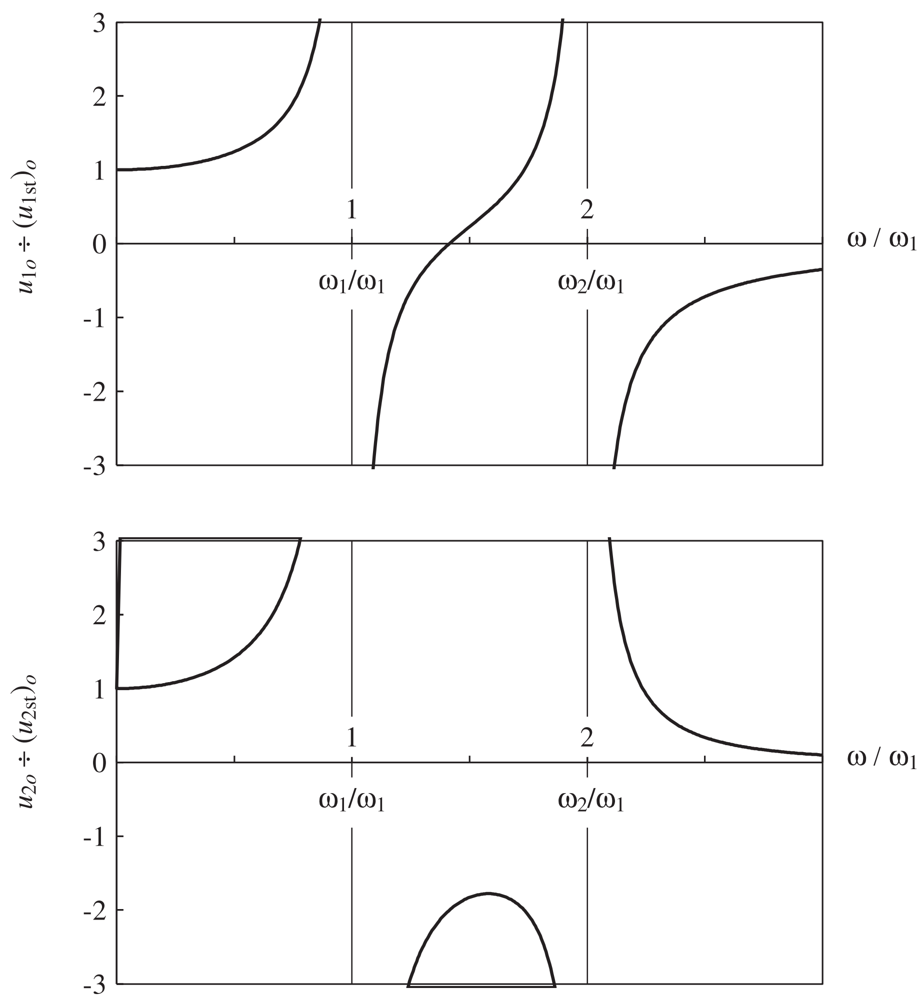

Figure E12.1

Figure E12.1 shows the normalized response amplitudes u_{1o} and u_{2o} plotted against the frequency ratio ω/ω_1. These frequency-response curves show two resonance conditions at $\omega = \omega_1$ and $\omega = \omega_2$; at these exciting frequencies the steady-state response is unbounded. At other exciting frequencies, the vibration is finite and could be calculated from Eq. (b). Note that there is an exciting frequency where the vibration of the first mass, where the exciting force is applied, is reduced to zero. This is the entire basis of the *dynamic vibration absorber* or *tuned mass damper* discussed next.

12.2 VIBRATION ABSORBER OR TUNED MASS DAMPER

The *vibration absorber* is a mechanical device used to decrease or eliminate unwanted vibration. The description *tuned mass damper* is often used in modern installation; this modern name has the advantage of showing its relationship to other types of dampers. In the brief presentation that follows, we restrict ourselves to the basic principle of a vibration absorber without getting into the many important aspects of its practical design.

In its simplest form, a vibration absorber consists of one spring and a mass. Such an absorber system is attached to a SDF system, as shown in Fig. 12.2.1a. The equations of motion for the main mass m_1 and the absorber mass m_2 are the same as Eq. (12.1.1). For harmonic force applied to the main mass we already have the solution given by Eq. (12.1.6). Introducing the notation

$$\omega_1^* = \sqrt{\frac{k_1}{m_1}} \qquad \omega_2^* = \sqrt{\frac{k_2}{m_2}} \qquad \mu = \frac{m_2}{m_1} \tag{12.2.1}$$

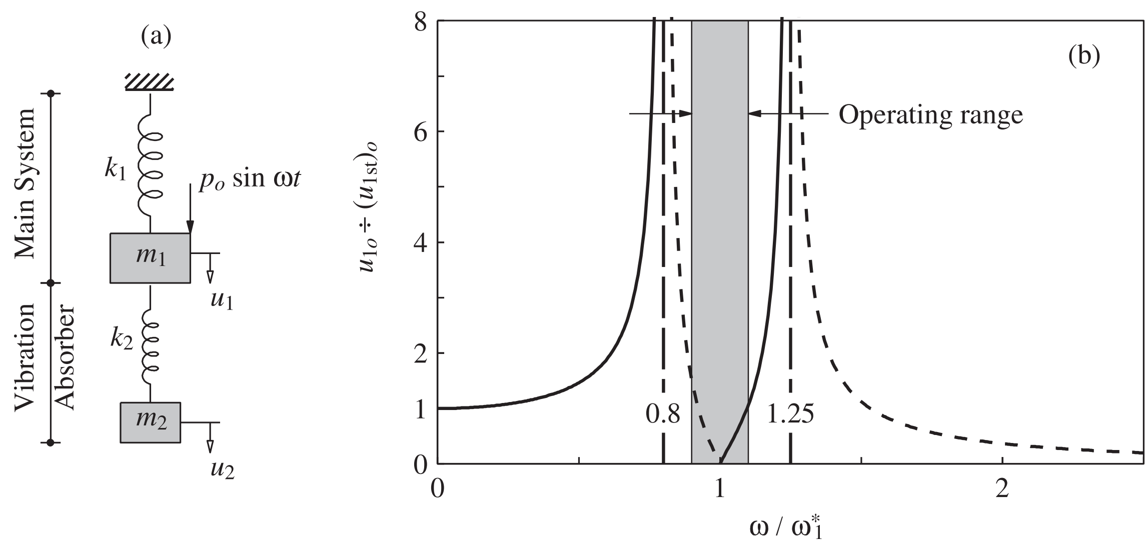

Figure 12.2.1 (a) Vibration absorber attached to an SDF system; (b) response amplitude versus exciting frequency (dashed curve indicates negative u_{1o} or phase opposite to excitation); $\mu = 0.2$ and $\omega_1^* = \omega_2^*$.

the available solution can be rewritten as

$$u_{1o} = \frac{p_o}{k_1} \frac{1 - \left(\omega/\omega_2^*\right)^2}{\left[1 + \mu\ \left(\omega_2^*/\omega_1^*\right)^2 - \left(\omega/\omega_1^*\right)^2\right]\left[1 - \left(\omega/\omega_2^*\right)^2\right] - \mu\ \left(\omega_2^*/\omega_1^*\right)^2} \tag{12.2.2a}$$

$$u_{2o} = \frac{p_o}{k_1} \frac{1}{\left[1 + \mu\ \left(\omega_2^*/\omega_1^*\right)^2 - \left(\omega/\omega_1^*\right)^2\right]\left[1 - \left(\omega/\omega_2^*\right)^2\right] - \mu\ \left(\omega_2^*/\omega_1^*\right)^2} \tag{12.2.2b}$$

At exciting frequency $\omega = \omega_2^*$, Eq. (12.2.2a) indicates that the motion of the main mass m_1 does not simply diminish, it ceases altogether. Figure 12.2.1b shows a plot of response amplitude $u_{1o} \div (u_{1\mathrm{st}})_o$, where $(u_{1\mathrm{st}})_o = p_o/k_1$, versus ω; for this example, the mass ratio $\mu = 0.2$ and $\omega_1^* = \omega_2^*$, the absorber being tuned to the natural frequency of the main system. Because the system has two DOFs, two resonant frequencies exist, and the response is unbounded at those frequencies. The operating frequency range where $u_{1o} \div (u_{1\mathrm{st}})_o < 1$ is shown.

The usefulness of the vibration absorber becomes obvious if we compare the frequency-response function of Fig. 12.2.1b with the response of the main mass alone, without the absorber mass. At $\omega = \omega_1^*$ the response amplitude of the main mass alone is unbounded but is zero with the presence of the absorber mass. Thus, if the exciting frequency ω is close to the natural frequency ω_1^* of the main system, and operating restrictions make it impossible to vary either one, the vibration absorber can be used to reduce the response amplitude of the main system to near zero.

What should be the size of the absorber mass? To answer this question, we use Eq. (12.2.2b) to determine the motion of the absorber mass at $\omega = \omega_2^*$:

$$u_{2o} = -\frac{p_o}{k_2} \tag{12.2.3}$$

The force acting on the absorber mass is

$$k_2 u_{2o} = \omega^2 m_2 u_{2o} = -p_o \tag{12.2.4}$$

This implies that the absorber system exerts a force equal and opposite to the exciting force. Thus, the size of the absorber stiffness and mass, k_2 and m_2, depends on the allowable value of u_{2o}. There are other factors that affect the choice of the absorber mass. Obviously, a large absorber mass presents a practical problem. At the same time the smaller the mass ratio μ, the narrower will be the operating frequency range of the absorber.

The preceding presentation indicates that a vibration absorber has its greatest application to synchronous machinery, operating at nearly constant frequency, for it is tuned to one particular frequency and is effective only over a narrow band of frequencies. However, vibration absorbers are also used in situations where the excitation is not nearly harmonic.

The dumbbell-shaped devices that hang from highest-voltage transmission lines are vibration absorbers used to mitigate the fatiguing effects of wind-induced vibration. Vibration absorbers have also been used to reduce the wind-induced vibration of tall buildings when the motions have reached annoying levels for the occupants. An example of this is the 59-story Citicorp Center in midtown Manhattan; completed in 1977, this building has a 820-kip block of concrete installed on the 59th floor in a movable platform connected to the building by large hydraulic arms. When the building sways more than 1 foot a second, the computer directs the arms to move the block in the other direction. This action reduces such oscillation by 40%, considerably easing the discomfort of the building's occupants during high winds.

PART B: MODAL ANALYSIS

12.3 MODAL EQUATIONS FOR UNDAMPED SYSTEMS

The equations of motion for a linear MDF system without damping were derived in Chapter 9 and are repeated here:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \tag{12.3.1}$$

The simultaneous solution of these coupled equations of motion that we have illustrated in Section 12.1 for a two-DOF system subjected to harmonic excitation is not efficient for systems with more DOFs, nor is it feasible for systems excited by other types of forces. Consequently, it is advantageous to transform these equations to modal coordinates, as we shall see next.

As mentioned in Section 10.7, the displacement vector $\mathbf{u}$ of an MDF system can be expanded in terms of modal contributions. Thus, the dynamic response of a system can be expressed as

$$\mathbf{u}(t) = \sum_{r=1}^{N} \phi_r q_r(t) = \mathbf{\Phi}\mathbf{q}(t) \tag{12.3.2}$$

Using this equation, the coupled equations (12.3.1) in $u_j(t)$ can be transformed to a set of uncoupled equations with modal coordinates $q_n(t)$ as the unknowns. Substituting Eq. (12.3.2) in Eq. (12.3.1) gives

$$\sum_{r=1}^{N} \mathbf{m}\,\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \mathbf{k}\,\phi_r q_r(t) = \mathbf{p}(t)$$

Premultiplying each term in this equation by ϕ_n^T gives

$$\sum_{r=1}^{N} \phi_n^T \mathbf{m} \phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T \mathbf{k} \phi_r q_r(t) = \phi_n^T \mathbf{p}(t)$$